

E_{11} : Sign of the times

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Abstract

We discuss the signature of space-time in the context of the E_{11} -conjecture. In this setting, the space-time signature depends on the choice of basis for the “gravitational sub-algebra” A_{10} , and Weyl transformations connect interpretations with different signatures of space-time. Also the sign of the 4-form gauge field term in the Lagrangian enters as an adjustable sign in a generalized signature. Within E_{11} , the combination of space-time signature (1,10) with conventional sign for the 4-form term, appropriate to M -theory, can be transformed to the signatures (2,9) and (5,6) of Hull’s M^* - and M' - theories (as well as (6,5), (9,2) and (10,1)). Theories with other signatures organize in orbits disconnected from these theories. We argue that when taking E_{11} seriously as a symmetry algebra, one cannot discard theories with multiple time-directions as unphysical. We also briefly explore links with the $SL(32, \mathbb{R})$ conjecture.

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1 Introduction

The discovery of duality symmetries has revolutionized the field of high energy physics. Many characteristics of a theory turn out to depend on one's point of view. Duality symmetries mix fields of different types, turn local excitations of a field into solitons, and connect theories in spaces of different topology, or even different dimensions [1, 2].

A fascinating, but puzzling discovery is that suitable duality symmetries of string and M-theory can even change the signature of space-time [3, 4, 5]. Although at the classical level a theory with more than one time-direction is as easily described as any other, it is not straightforward to extend this to the quantum regime. Moreover, the construction of [3, 4] involves the compactification of a *time-like* direction and therefore inevitably leads to closed time-like curves, with all the problems associated to them. Therefore, even though it is clear that the ideas of [3, 4] are correct and useful at the mathematical level, it is not clear what physical significance should be attached to them.

It is believed that the ultimate fusion of the covariance properties of the elf-bein of 11 dimensional supergravity [6], with the exceptional E -type duality symmetries of compactified maximal supergravity [7, 8, 9], can be achieved within a huge Kac-Moody algebra, which is called E_{11} [10]. For the E_n groups with $n < 11$ that are associated to dimensional reduction to $11 - n$ dimensions, we may always choose the time-direction to be transverse to the dimensions we are reducing over (although it can be included [11]), but for E_{11} this is no longer possible: We have to deal with the imprint that a time direction makes on the algebra.

The conjecture states that E_{11} is realized non-linearly, as a coset symmetry E_{11}/H_{11} . Technically, as we will explain in section 3, the signature of space-time can be incorporated by introducing a number of signs in the denominator sub-algebra H_{11} [12, 13] (these signs also play a crucial role in the computations in [14]). The H_{11} algebra relations imply that such signs will proliferate throughout the algebra. According to [15, 16, 17] (see also [18]) the matter content of the theory can be recovered by making a level decomposition with respect to a “horizontal” $A_{10} = sl(11, \mathbb{R})$ sub-algebra. The signs that were introduced in the gravity sector, imply signs in the sectors corresponding to the gauge fields, and beyond. Dual descriptions of the theory correspond to alternative choices for the A_{10} algebra. There is no reason to expect that all choices for A_{10} will lead to the same signature of space-time, and indeed this turns out not to be the case. The possibility to exchange signs between the gravity and gauge sector can be seen as the algebraic explanation for signature-changing dualities.

This paper studies the imprint of the space-time signature (1,10) on the E_{11} algebra (throughout this paper we will denote the space-time signature as (t, s) , with t the number of time-like directions, and s the number of space-like directions). Our main message will be that the version of E_{11}/H_{11} that contains the theories with signature (1,10) also

contains the signatures (2,9), (5,6),(6,5), (9,2), and (10,1). We therefore argue that if one takes E_{11} seriously as a symmetry algebra, one cannot discard the M^* - and M' - theories as unphysical; they come for free with the algebra.

In absence of certain physical requirements, such as supersymmetry, all signatures are possible: The bosonic sector that is encoded in E_{11} does not impose any restrictions. We study how all possible signatures connect under the Weyl group of E_{11} . We emphasize that in absence of requirements as signature (1,10), or a definite sign for the 4-form gauge-field term (requirements which we also have to give up for M-theory as it seems), these correspond to acceptable theories. They however cannot be supersymmetrized.

In this paper we introduce some techniques, and work them out very explicitly. The mathematically inclined reader will notice that these techniques allow some level of abstraction, and are by no means limited to E_{11} . In a subsequent paper [19] we will apply these techniques to elucidate the full (space- and time-like) duality web for M-theory and its cousins, to find among other things a few duality groups that seem to have gone unnoticed in previous works.

2 Definition and properties of E_{11}

In this section we recall some facts about the general theory of Kac-Moody algebra's [20], and E_{11} in particular. We start by drawing the Dynkin diagram of E_{11} .

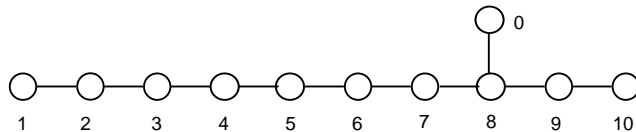


Figure 1: The Dynkin diagram of E_{11} .

From this diagram the Cartan matrix $A = (a_{ij})$, with i, j in the index set $I \equiv \{0, 1, \dots, 10\}$, may be reconstructed by setting

$$a_{ij} \equiv \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i, j \text{ connected by a line;} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

The Cartan Matrix is symmetric, and $\det(A) = -2$. We choose a real vector space \mathcal{H} of dimension 11 and linearly independent sets $\Pi = \{\alpha_0, \dots, \alpha_{10}\} \subset \mathcal{H}^*$ (with \mathcal{H}^* the space dual to \mathcal{H}) and $\Pi^\vee = \{\alpha_0^\vee, \dots, \alpha_{10}^\vee\} \subset \mathcal{H}$, obeying $a_{ij} = \alpha_j(\alpha_i^\vee) \equiv \langle \alpha_j, \alpha_i^\vee \rangle$. The elements of the set Π are called the simple roots.

From the Cartan matrix the algebra E_{11} can be constructed. The generators of the algebra consist of 11 basis elements h_i for the Cartan sub algebra \mathcal{H} together with 22 generators e_{α_i} and $e_{-\alpha_i}$ ($i \in I$), and of algebra elements obtained by taking multiple commutators of these. These commutators are restricted by the algebraic relations (with $h, h' \in \mathcal{H}$):

$$[h, h'] = 0 \quad [h, e_{\alpha_j}] = \langle \alpha_j, h \rangle e_{\alpha_j} \quad [h, e_{-\alpha_j}] = -\langle \alpha_j, h \rangle e_{-\alpha_j} \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \alpha_i^\vee; \quad (2)$$

and the Serre relations

$$\text{ad}(e_{\alpha_i})^{1-a_{ij}} e_{\alpha_j} = 0, \quad \text{ad}(e_{-\alpha_i})^{1-a_{ij}} e_{-\alpha_j} = 0. \quad (3)$$

Implicit in the conjecture is that one is working with a real form of E_{11} , to be precise the split real form, which is generated by linear combinations of the above generators with *real* coefficients.

As is well known, the Cartan matrix of E_{11} , implies that the root space is of signature (10,1). Consider the space defined by 11-tuples forming vectors $p = (p_1, \dots, p_{11}) \in \mathbb{R}^{11}$, with norm

$$||p||^2 = \sum_{i=1}^{11} p_i^2 - \frac{1}{9} \left(\sum_{i=1}^{11} p_i \right)^2, \quad (4)$$

and inner product \langle, \rangle defined by the norm via

$$\langle a, b \rangle = \frac{1}{2} (||a+b||^2 - ||a||^2 - ||b||^2) = \sum_{i=1}^{11} a_i b_i - \frac{1}{9} \left(\sum_{i=1}^{11} a_i \right) \left(\sum_{i=1}^{11} b_i \right). \quad (5)$$

With this inner product, the signature of the root space is (10,1): the 10 dimensional subspace of vectors x defined by $\sum_{i=1}^{11} x_i = 0$ contains only vectors of positive norm; the 1 dimensional orthogonal complement consisting of vectors x of the form $x_i = \lambda, \lambda \in \mathbb{R}, \forall i$ has vectors of negative norm (these choices were inspired by the choice of metric on the E_{10} root space in [21, 22]).

We realize the simple roots of E_{11} in the above space as

$$\alpha_i = e_i - e_{i+1}, \quad i = 1, \dots, 10; \quad (6)$$

$$\alpha_0 = e_9 + e_{10} + e_{11}. \quad (7)$$

Here $(e_i)_j = \delta_{ij}$, and the reader should notice that with the inner product (5) these are *not* unit vectors. The root lattice of E_{11} , which we will call P_{11} , consists of linear combinations of the simple roots, with coefficients in \mathbb{Z} . This lattice can be characterized as consisting of those vectors a whose components a_i are integers, and sum up to three-folds $3k$. The integer k counts the occurrences of the “exceptional” root α_0 , and for roots equals the level as defined in [15, 16] (see also [18]).

Via the innerproduct defined as above, and the relations $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ it follows that there is a natural bijection $\alpha_i \rightarrow \alpha_i^\vee$, in which the components of each α_i^\vee turn out to

be identical to those of α_i (This is because E_{11} is a simply-laced algebra). The coroot lattice, consisting of linear combinations with integer coefficients of α_i^\vee may therefore be identified with the root lattice. In view of this, our two different definitions of \langle, \rangle are numerically equivalent.

There is the root space decomposition with respect to the Cartan subalgebra, $E_{11} = \oplus_{\alpha \in \mathcal{H}^*} g_\alpha$, with

$$g_\alpha = \{x \in E_{11} : [h, x] = \langle \alpha, h \rangle x \ \forall h \in \mathcal{H}\} \quad (8)$$

The set of roots of the algebra, Δ , are defined by

$$\Delta = \{\alpha \in \mathcal{H} : g_\alpha \neq 0, \alpha \neq 0\} \quad (9)$$

The roots of the algebra form a subset of the root lattice $\Delta \subset P_{11}$. The set of positive roots $\Delta^+ \subset \Delta$ is the subset of roots whose expansion in the simple roots involves non-negative integer coefficients only. We denote the basis elements of g_α by e_α^k , where k is a degeneracy index, taking values in $\{1, \dots, \dim(g_\alpha)\}$. If $\dim(g_\alpha) = 1$, we will drop the degeneracy index, and write e_α for the generator. This is in accordance with previous notation, as $\dim(g_{\alpha_i}) = 1$ when α_i is a simple root. By using the Jacobi identity, one can easily prove that $[e_\alpha^i, e_\beta^j] \in g_{\alpha+\beta}$, if this commutator is different from zero.

The inverse Cartan matrix is given by (notice the sign in front):

$$-A^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 0 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 & 12 & 15 & 18 & 12 & 6 & 9 \\ 2 & 4 & 6 & 8 & 12 & 16 & 20 & 24 & 16 & 8 & 12 \\ 3 & 6 & 9 & 12 & 15 & 20 & 25 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 24 & 30 & 36 & 24 & 12 & 18 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 42 & 28 & 14 & 21 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 32 & 16 & 24 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 20 & 10 & 16 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 10 & 4 & 8 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 16 & 8 & 11 \end{pmatrix} \quad (10)$$

The fundamental coweights ω_i are defined by

$$\langle \alpha_i, \omega_j \rangle = \delta_{ij}. \quad (11)$$

They can be expressed in the coroots as $\omega_i = (A^{-1})_{ij}(\alpha_j)^\vee$. In the basis we have chosen,

the explicit form of the fundamental coweights is:

$$\begin{aligned}
-\omega_1 &= \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\
-\omega_2 &= \left(0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \right) \\
-\omega_3 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \\
-\omega_4 &= \left(1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2 \right) \\
-\omega_5 &= \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2} \right) \\
-\omega_6 &= \left(2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3 \right) \\
-\omega_7 &= \left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}, \frac{7}{2} \right) \\
-\omega_8 &= \left(3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4 \right) \\
-\omega_9 &= \left(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3 \right) \\
-\omega_{10} &= \left(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2 \right) \\
-\omega_0 &= \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right)
\end{aligned} \tag{12}$$

The coweight lattice, which we call Q_{11} consist of linear combinations of the fundamental coweights with coefficients in \mathbb{Z} . It can be characterized as those vectors that have components that are either all integers, or all odd integers divided by 2, and whose components sum up to multiples of $\frac{3}{2}$. It is clear that the coweight lattice contains the coroot lattice, which is an index 2 sublattice.

We will make much use of the Weyl group W_{11} of E_{11} . This is the group generated by the Weyl reflections w_i in the simple roots,

$$w_i(\beta) = \beta - 2 \frac{\langle \alpha_i, \beta \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i \tag{13}$$

The Weyl group leaves the inner product invariant

$$\langle w(\alpha), w(\beta) \rangle = \langle \alpha, \beta \rangle \quad w \in W_{11} \tag{14}$$

The Weyl group includes reflections in the non-simple roots. In the basis we have chosen the Weyl reflections w_i in α_i , $i = 1, \dots, 10$ permute entries of the row of 11 numbers. Crucial to our story are the Weyl reflections in α_0 , and other roots that have α_0 exactly once in their expansion. We will need these often, and find it convenient to introduce specific notation to represent these roots. They are all of the form

$$\beta_{ijk} = e_i + e_j + e_k, \quad i < j < k$$

For example $\alpha_0 = \beta_{91011}$

3 Space-time signature and U-duality algebra

After the many technicalities introduced in the previous section, we proceed with explaining in what way they enter the discussion on physics.

3.1 Compact and non-compact generators of H_{11}

The conjectured E_{11} -symmetry of M -theory is non-linearly realized [10]. The relevant variables are not described by E_{11} , but by the coset E_{11}/H_{11} . By analogy to the cosets appearing in dimensional reduction (see e.g. [23, 24] and also [18]), it is proposed that the degrees of freedom can be retrieved from E_{11} by decomposing the algebra with respect to a horizontal A_{10} algebra [15, 16, 17]. The real form corresponding to $A_{10} = SL(11, \mathbb{R})$; the 11th generator of the Cartan subalgebra (generating a non-compact scaling symmetry isomorphic to \mathbb{R}) can be added to this to give the algebra of $GL(11, \mathbb{R})$. This is argued to be the $GL(11, \mathbb{R})$ relevant to the local vielbein, which can be regarded as an element of $GL(11, \mathbb{R})/SO(1, 10)$. Hence, when specifying the real form of H_{11} , we will start with the horizontal subalgebra $SO(1, 10)$.

The maximal compact subgroup inside E_{11} can be defined as the group invariant under the Cartan involution; it consists of generators of the form $e_\alpha^k - e_{-\alpha}^k$. In our conventions $(e_\alpha^k)^\dagger = e_{-\alpha}^k$, and therefore $e_\alpha^k - e_{-\alpha}^k$ is anti-hermitian, implying that it generates a compact symmetry. It was observed in [12, 13] (see also [14] for an application) that the possibility of non-compact denominator groups can be taken into account by including a sign $\epsilon_\alpha = \pm 1$ in the generator, such that the real form of H_{11} will be generated by generators of the form

$$T_\alpha^k = e_\alpha^k - \epsilon_\alpha e_{-\alpha}^k, \quad \alpha \in \Delta^+ \quad (15)$$

Because both α and $-\alpha$ enter in the definition, we can restrict to $\alpha \in \Delta^+$. We will explain in a moment that ϵ_α cannot depend on the degeneracy index k , but we have already omitted it from our notation.

For $\epsilon_\alpha = 1$ T_α^k is anti-hermitian, whereas for $\epsilon_\alpha = -1$ it is hermitian. Now it is a simple fact that for H, H' hermitian, and A, A' anti-hermitian,

$$[H, H'], \text{ and } [A, A']$$

are anti-hermitian, whereas

$$[H, A]$$

is hermitian. Using that the structure constants of E_{11} , and hence of H_{11} are real, we deduce that the hermiticity properties of the generators $T_{\alpha \pm \beta}^k$ follow directly from those of T_α^k and T_β^k . Upon using that all elements T_α^k can be formed from taking multiple commutators of the T_{α_i} , with α_i the simple roots (where we can again drop the degeneracy index k), we see that ϵ_α does depend on α , but cannot depend on the degeneracy index.

We now form a \mathbb{Z}_2 -valued linear function $f(\alpha)$ on the root lattice, by defining for the simple roots

$$f(\alpha_i) = 0 \quad \text{if } (T_{\alpha_i})^\dagger = -T_{\alpha_i} \quad (16)$$

$$f(\alpha_i) = 1 \quad \text{if } (T_{\alpha_i})^\dagger = T_{\alpha_i} \quad (17)$$

and continue on the root lattice by linearity. If α belongs to the root lattice of E_{11} , then $f(\alpha)$ specifies whether the corresponding generator(s) T_α^i of H_{11} are compact or non-compact generators: $f(\alpha) = 0$ implies anti-hermiticity, and therefore compactness of the symmetry generated by this generator, while $f(\alpha) = 1$ implies hermiticity, and non-compactness of the generator. Consequently, the function f specifies the real form of the denominator subgroup H_{11} .

It is important to realize that, even though H_{11} consists of infinitely many elements, that there are only finitely many \mathbb{Z}_2 -valued linear functions on the root lattice. These are completely specified by their values on the simple roots, so there are only $2^{11} = 2048$ such functions. It is actually easy to describe them, they are all of the form

$$f(\alpha) = \sum_i p_i \langle \alpha, \omega_i \rangle \bmod 2, \quad \alpha \in P_{11}, \quad (18)$$

where, as $\langle \alpha, \omega_i \rangle \in \mathbb{Z}$, the coefficients p_i can be chosen to be in \mathbb{Z}_2 . Below we will specify the explicit entries of f as a row of 11 numbers. The mod 2 property of the function f then translates to equivalence under shifts by elements of $2Q_{11}$ (which is a sublattice of the coroot lattice, which we have identified with the root lattice P_{11}).

The function f is completely specified by its values on a basis of simple roots. There are however many such bases. We regard two choices of basis for the root lattice as equivalent if they are related by a sequence of Weyl reflections, that is by an element of the Weyl group. If the original basis is given by $\{\alpha_i\}_{i \in I}$, then the Weyl transformed basis is given by $\{w(\alpha_i)\}_{i \in I}$, $w \in W_{11}$. The transformation w does not alter H_{11} , and hence at the mathematical level describes the same theory. It may however rotate signs from the gravity into the gauge sector, and vice versa, and hence the interpretation of the mathematical theory may result in a different space-time signature and/or a different sign in front of the 4-form term.

As, due to the property (14) of the Weyl group

$$\langle w(\alpha_i), f \rangle = \langle \alpha_i, w^{-1}(f) \rangle, \quad w, w^{-1} \in W_{11}, \quad (19)$$

the theory with the Weyl rotated basis and function f , gives the same theory as the original basis with a Weyl rotated f . It is however easier to study the action on one linear function f than on a basis of 11 roots α_i . Finding all possible non-compact forms that can appear in the denominator group therefore amounts to classifying all functions f up to Weyl reflections.

We have transcribed our problem to a mathematical setting. Before turning to E_{11} we need to sort one more thing out: The relation between the function f and the signature of space-time.

3.2 Space-time signature from H_{11}

To figure out what the space-time signature corresponding to a certain function f is, we have to study its values on the A_{10} sub-algebra that defines the gravitational sector of the theory. Again it will suffice to study its value on the simple roots, but it is instructive to check the action on the full set of roots (which for A_{10} , in contrast to E_{11} , is a finite set, and therefore manageable).

The 55 positive roots corresponding to the A_{10} subalgebra are given by

$$\sum_{i=k}^l \alpha_i, \quad 1 \leq k \leq l \leq 10. \quad (20)$$

It will be clear that f and $f + \omega_0$ correspond to the same signature of the A_{10} algebra, so for the time being we will ignore ω_0 . We will return to its significance later.

Consider $f = \omega_1$. This gives 1 on all roots having α_1 in its expansion. More precisely, it gives 1 on all roots in which the root α_1 appears an odd number of times, but in all roots of A_{10} it appears at most once. There are 10 positive roots of A_{10} containing α_1 , and 45 which do not contain it. This means that 10 out of the T_α generators, will be non-compact generators, while the other 45 are compact ones. Knowing that the algebra generated by the T_α is a real form of $so(11, \mathbb{C})$, we immediately identify the algebra as the one of $so(10, 1)$.

As another example, set $f = \omega_2$. The same procedure as above reveals that there are 18 non-compact generators, and 37 compact ones. Hence the algebra must be $so(2, 9)$. Similarly, it is easy to verify that $f = \omega_p$ corresponds to $so(p, 11 - p)$.

To consider a more complicated example, take $f = \omega_1 + \omega_3$. There are 10 roots containing α_1 , and 24 containing α_3 . There are however 8 roots that contain α_1 as well as α_3 , and that are therefore mapped to 0 mod 2 by f . There are therefore 18 non-compact generators, and again the denominator sub-algebra is $so(2, 9)$. As a matter of fact, the coweight $\omega_1 + \omega_3$ is equivalent to ω_2 , under a suitable combination of Weyl reflections and translations over elements from $2Q_{11}$.

A less abstract and more direct way to see the group is the following. Let $(x_1, x_2, \dots, x_{11})$ be the coordinates on the eleven dimensional tangent space (on which the denominator subgroup of A_{10} acts). Take T_{α_i} ($1 \leq i \leq 10$) to be the generator that mixes the x_i and x_{i+1} . Now T_{α_i} generates $SO(2)$ if it is anti-hermitian, and $SO(1, 1)$ if it is hermitian. Therefore if $f(\alpha_i) = 0$, then x_i and x_{i+1} correspond both to space-, or both to time-like directions, whereas when $f(\alpha_i) = 1$, one of them is a space-like and the other one a time-like direction. Evaluating f successively on α_{10} , α_9 , etc. immediately gives the signature.

This can be summarized in the following simple procedure: To compute the denominator

group in A_{10} corresponding to f , write f as the sum of n weights, and order these as

$$\omega_{k_1} + \omega_{k_2} + \dots + \omega_{k_n}, \quad 10 \geq k_1 > k_2 > \dots > k_n \geq 0.$$

Then the algebra is given by

$$so(k, 11 - k); \quad k = \sum_{i=1}^n (-1)^{i+1} k_i. \quad (21)$$

Note that $k \geq 0$, and ω_0 does not contribute.

Having found that the signature of space-time is encoded in the weights $\omega_1, \dots, \omega_{10}$, what is the significance of ω_0 ? There is yet another source of minus signs in the algebra, coming from the sign in front of the $*G \wedge G$ term in the 11-dimensional Lagrangian, corresponding to the 3-form gauge field. A certain choice of space-time signature implies that there are all kinds of signs implicit in the contraction of this term. We can however also adjust the overall sign in front of this term. It is precisely this minus-sign that is encoded in ω_0 . That it works this way can actually be easily seen by comparing to the sigma-model action for the effective theories representing toroidal compactification of less than 11 dimensions [25, 24], but we will discuss this more completely elsewhere [19].

We define a “generalized signature” (t, s, \pm) , denoting by t the number of time directions, by s the number of space-like direction, and \pm giving the sign in front of the 4-form gauge-field term, relative to the conventional one. In terms of this sign, the bosonic part of the 11 dimensional Lagrangian equals

$$R - (\pm *G \wedge G), \quad (22)$$

plus the Chern-Simons term, that is of no relevance in this discussion, as its sign can be changed by redefining the 3-form potential $C_{(3)} \rightarrow -C_{(3)}$, which is a transformation without any relation to space-time.

The discussion in the above gives the relative signature $|t - s|$, we cannot distinguish between signature (t, s) and (s, t) . The $\binom{11}{3} = 165$ combinations $T_\alpha = e_\alpha - \epsilon_\alpha e_{-\alpha}$, for which α_0 enters with multiplicity 1 in the expansion of α , are loosely interpreted as associated to the components of the 3-form potential giving rise to the 4-form field strength [15, 16]. In a space-time with signature (t, s) , the number of components of the 3 form with an odd number of space-like directions is

$$\binom{t}{0} \binom{s}{3} + \binom{t}{2} \binom{s}{1} \quad (23)$$

(where we put binomials $\binom{p}{q}$ that would be ill-defined, that is when $q > p$, to zero). These correspond to compact generators. On the other hand, the number of components of the 3-form with an even number of space-like dimensions (giving non-compact generators), is

$$\binom{t}{1} \binom{s}{2} + \binom{t}{3} \binom{s}{0}. \quad (24)$$

Interchanging $t \leftrightarrow s$ interchanges (23) with (24). Hence, if we denote a function f , representing t time directions, s space-directions, and a conventional sign in front of the 4-form term by $f_{(t,s,+)}$, then there is an equality

$$f_{(t,s,+)} = f_{(s,t,-)} \quad (25)$$

where $f_{(s,t,-)}$ is a function representing a theory with s time directions, t space-directions, and an unconventional sign in front of the 4-form term.

Theories with generalized signature $(t, s, +)$ and $(s, t, -)$ exist, and are described by the same function. To argue that they can be connected by Weyl transformations, we must look at the tangent space with coordinates (x_1, \dots, x_{11}) and work out the corresponding relative signature. Then we *choose* the absolute signature, by assigning that one group of coordinates describes space-like, and the other group time-like directions. If we now fix a group of space-like (time-like) coordinates, and act with Weyl transformations on generators that preserve these coordinates, we can reach other theories, where the group of fixed coordinates still describes space-like (time-like) directions. Doing this in successive steps often (but not always) the theory with space-time signature (s, t) can be reached from the theory with signature (t, s) . This is essentially the algebraic transcription of the procedure followed in [4] to deduce the existence of the theory with space-time signature (s, t) from the one of signature (t, s) , as in the toroidal context, the Weyl reflection in β_{ijk} corresponds to choosing a 3-torus, and rewriting to the theory for the dual 3-torus.

4 E_{11} and the signature of space-time

By itself, E_{11} does not prefer any signature of space-time. As should have become clear in the previous, we are free to choose any space-time signature, as well as the overall sign for the 4-form term in the Lagrangian. We can, and will work out all possible signatures and theories, but emphasize that only a subset of them meets certain physical requirements, such as compatibility with supersymmetry.

4.1 Space-time signatures compatible with M -theory

We will start with the algebra that represents the bosonic sector of the conventional 11 dimensional supergravity [6], having signature $(1, 10)$ and a conventional sign in front of the 4-form term.

Such a theory can be represented by the function $f_{(1,10,+)}$, given by

$$f_{(1,10,+)} = f_{(10,1,-)} = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad (26)$$

which is actually $-\omega_1$, but under shifts by $2Q_{11}$ we do not need to worry about the overall sign. There are $\binom{11}{1} = 11$ permutations of the entries, which give different functions, but do of course correspond to the same space-time signature (and 4-form term).

To find another space-time signature, described by the same non-compact form of H_{11} , we apply a Weyl reflection in β_{123} . After permuting the entries we arrive at

$$f_{(2,9,-)} = f_{(9,2,+)} = \left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad (27)$$

By permutations there are $\binom{11}{2} = 55$ choices that give the same space-time signature.

We have set the third index on $f_{(2,9,-)}$ to $(-)$. If we assume a space-time signature $(2, 9)$ the 165 components of the 3-form gauge-field divide into 93 terms with positive sign, giving compact generators, and 72 of the opposite sign, giving non-compact generators. It is however easily verified that $f_{(2,9,-)}$ results in 72 compact generators in the 3-form sector, and 93 non-compact ones. Therefore we have to invoke the extra minus sign in front of the 4-form field strength to make the correspondence work. Note that this extra minus sign is also crucially present in M^* -theory [4].

To find yet another space-time signature, we apply a Weyl reflection to $f_{(2,9,-)}$ in β_{345} , to arrive at

$$f_{(5,6,+)} = f_{(6,5,-)} = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad (28)$$

By permuting the entries there are $\binom{11}{5} = 462$ choices that give the same space-time signature. Again the reader should take notice of the $+$ appearing on $f_{(5,6,+)}$, this time the signs in the 4-form field terms are in accordance with the space-time signature (which again agrees with the findings of [4]).

Now let (28) correspond to a theory for which the coordinates x_1, \dots, x_4 correspond to time-like directions, whereas the coordinates x_8, \dots, x_{11} correspond to space-like directions. Then the space-time signature is $(5, 6)$. Weyl reflecting in β_{567} and making suitable shifts over $2Q_{11}$, one arrives at

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right), \quad (29)$$

which is, up to permutations of its entries equivalent to $f_{(5,6,+)}$, but under the identifications we have made for $x_1, \dots, x_4, x_8, \dots, x_{11}$ it must represent a theory with space-time signature $(6, 5)$.

Continuing from here we can reach all of the signatures $(1, 10)$, $(2, 9)$, $(5, 6)$, $(6, 5)$, $(9, 2)$, $(10, 1)$. Further Weyl reflections to $f_{(1,10,+)}$, $f_{(2,9,-)}$ and $f_{(5,6,+)}$ do not generate new functions; all are, up to shifts by $2Q_{11}$ and permutation of the entries equivalent to $f_{(1,10,+)}$, $f_{(2,9,-)}$ and $f_{(5,6,+)}$. We conclude that these $11 + 55 + 462 = 528$ \mathbb{Z}_2 -valued functions on the E_{11} root lattice form a closed orbit under Weyl reflections. They correspond to

choices of signs relevant to M -, M^* - and M' -theory [4]. We note that it is essentially the Weyl reflection in the exceptional root α_0 that allows for space-time signature changing transformations.

4.2 Other space-time signatures

There are three more, inequivalent orbits of the Weyl group, acting on the \mathbb{Z}_2 -valued functions defining various space-time signatures. If we accept the existence of theories with exotic space-time signatures, and wrong signed 4-form terms (as the E_{11} conjecture, and time-like T-duality seem to impose on us), then these other theories are acceptable too. They however cannot be supersymmetrized, as spinors in these signatures do not have the right number of components to produce a supersymmetry algebra (see [4]).

A theory that is straightforwardly interpreted is the one corresponding to

$$f_{(0,11,+)} = f_{(11,0,-)} = 0, \quad (30)$$

corresponding to the compact form of the denominator algebra H_{11} . It is clear that this single entry fills out a whole orbit by itself. The “physical” interpretation gives a theory on a Euclidean space-time, with the conventional sign for the 4-form gauge field. This version may have its usual importance as representing the symmetries of the Wick-rotated $(1, 10)$ theory. There is also a version with time-like directions only, and an unconventional sign of the 4-form gauge-field term. This theory *cannot* be reached from the Euclidean theory.

We move on to the next class of theories. A convenient starting point is the algebraic realization of the bosonic sector of 11-d supergravity, on a Euclidean space-time, but now with the wrong sign on the 4-form term in the action. From our previous discussion it should be clear that this theory can be represented by

$$f_{(0,11,-)} = f_{(11,0,+)} = -\omega_0 = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right). \quad (31)$$

After a Weyl reflection in β_{123} this turns into

$$f_{(3,8,+)} = f_{(8,3,-)} = \left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right). \quad (32)$$

There are $\binom{11}{3} = 165$ permutations that give the same space-time signature.

A Weyl reflection in β_{145} , permutations of the entries, and subsequent shifts over $2Q_{11}$ take us to

$$f_{(4,7,-)} = f_{(7,4,+)} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right), \quad (33)$$

which gives, by permutation of the entries, $\binom{11}{4} = 330$ choices with the same space-time signature.

It is a simple exercise to show that theories of signature $(4, 7)$ and $(7, 4)$ are connected by Weyl reflections. Further Weyl reflections always take us back to $f_{(11,0,-)}$, $f_{(3,8,+)}$ and $f_{(4,7,-)}$, and functions equivalent to these. Hence this orbit is closed, and contains $1 + 165 + 330 = 496$ \mathbb{Z}_2 -valued functions. It is amusing to see that it precisely contains those space-time signatures not allowed by 11-dimensional supersymmetry.

It turns out there is one remaining orbit. A convenient starting point is the theory with signature $(1, 10)$, but the wrong sign for the 4-form term. This is described by

$$f_{(1,10,-)} = f_{(10,1,+)} = -\omega_{10} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2). \quad (34)$$

A Weyl reflection in β_{123} takes us to

$$f_{(4,7,+)} = f_{(7,4,-)} = (2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1). \quad (35)$$

A Weyl reflection in β_{156} , followed by permutation of the entries and a suitable shift over $2Q_{11}$ takes us to

$$f_{(5,6,-)} = f_{(6,5,+)} = (1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0). \quad (36)$$

A Weyl reflection over β_{456} takes us to

$$f_{(8,3,+)} = f_{(3,8,-)} = (1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0). \quad (37)$$

And a final Weyl reflection in β_{234} followed by suitable permutations takes us to

$$f_{(9,2,-)} = f_{(2,9,+)} = (1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0). \quad (38)$$

Notice that this function allows 110 permutations of the entries, but that as f and $-f$ represent the same signs, these only correspond to $\binom{11}{2} = 55$ \mathbb{Z}_2 -valued functions. This last orbit contains $\binom{11}{1} + \binom{11}{2} + \binom{11}{3} + \binom{11}{4} + \binom{11}{5} = 1023$ \mathbb{Z}_2 -valued functions.

Detailed examination of specific signatures reveals that the theories with space-time signatures $(1, 10)$, $(4, 7)$, $(5, 6)$, $(8, 3)$ and $(9, 2)$ are connected by Weyl reflections. Also the theories with signatures $(10, 1)$, $(7, 4)$, $(6, 5)$, $(3, 8)$ and $(2, 9)$ are connected, but it is impossible to reach say $(1, 10, -)$ from $(10, 1, +)$. Hence, like the function specifying the Euclidean theory, this single orbit of functions actually specifies 2 orbits of theories.

We have now listed one representative for each possibility for the generalized space-time signature. Moreover, as

$$528 + 1 + 496 + 1023 = 2048,$$

we have succeeded in dividing all \mathbb{Z}_2 -valued functions on the root lattice of E_{11} into 4 orbits of its Weyl group. As we have explained these correspond to 6 groups of theories. These results imply that there are 4 possible real forms of H_{11} that can appear in E_{11}/H_{11} , of which one corresponds to choices of signs appropriate for M -, M^* - and M' -theories.

5 Relation to the $SL(32)$ -conjecture

In [26] (see also [27]) it was observed that the $sl(32, \mathbb{R})$ algebra gives a formal symmetry of the 11-dimensional supercovariant derivative, and it was argued that this algebra should provide a useful tool in the classification of solutions. Proposals to elevate $SL(32, \mathbb{R})$ to a local symmetry of M -theory however turn out to be problematic with the representations present in 3-dimensional supergravity [28], and appears to be incompatible with the most general form of the E_{11} conjecture, as the local subgroup in E_{11} does not have an $SL(32, \mathbb{R})$ subgroup [29, 13].

There is however a remarkably simple relation between E_{11} and $SL(32, \mathbb{R})$, which was outlined in [29] and completed and generalized in [13]. The algebra $sl(32, \mathbb{R})$ can be obtained by truncating the local subalgebra H_{11} in a specific way. The procedure essentially amounts to truncating the E_{11} algebra to exclude the generators for which the absolute value of the level k exceeds 4, and then removing all generators from the remaining ones in $H_{11} \subset E_{11}$ that do not fall into antisymmetric tensor representations of the horizontal subalgebra $so(1, 10)$, which is a real form of B_5 .

Without further discussion on the validity and merit of the $SL(32, \mathbb{R})$ conjectures, we observe that our arguments as developed for E_{11} immediately allow an easy determination of the subgroup surviving the above truncation procedure. The generators at level 0 form the adjoint **55** of B_5 , the generators at level ± 1 combine into the 3-form **165**, the generators at level ± 2 organize in a 6-form **462**, of the generators at level ± 3 a 7-form **330** survives, and of the generators at level ± 4 a 10-form **11** remains, for a total of 1023 generators, which make up the Lie algebra of A_{31} [26].

We have to divide these into groups of hermitian and anti-hermitian generators. As these correspond to non-compact and compact generators, subtracting the number of the anti-hermitian generators from the number of the hermitian generators immediately gives the signature of the real form of A_{31} (the reader should not confuse the signature of the algebra with the space-time signature of the corresponding theory).

It is perhaps most convenient to start with the Euclidean theory, with conventional sign for the 4-form. The truncation as described above leads to 1023 compact generators, hence a real form of A_{31} with signature -1023 . The corresponding algebra must be $su(32)$.

The theory of space-time signature $(1, 10)$, with conventional sign for the 4-form gauge field gives 527 hermitian generators, against 496 compact ones. Therefore the signature of the algebra is 31, reproducing the well-known $sl(32, \mathbb{R})$.

The theory in Euclidean signature but with wrong signed 4-form term gives anti-hermitian generators for the generators from level 0, 2 and 4, and hermitian for the ones from level 1 and 3. This gives 528 compact generators, against 495 non-compact ones, and fixes the signature of the algebra to -33 . We therefore identify the algebra as the one of $su^*(32)$.

The theory of space-time signature $(1, 10)$ with unconventional sign for the 4-form term results in 511 compact generators, versus 512 non-compact ones, giving the value 1 for the signature of the algebra, which identifies the algebra as $su(16, 16)$.

We note that the real form of A_{31} is specified by precisely the same ingredients as the real form of H_{11} , namely by specifying the hermiticity properties of $\gamma_{i,i+1} \in A_{31}$ (corresponding to $T_{\alpha_i} \in H_{11}$) and the hermiticity properties of $\gamma_{9,10,11} \in A_{31}$ (which corresponds to $T_{\alpha_0} \in H_{11}$) (compare with the arguments of [13]). Furthermore, the argument on the commutators from subsection 3.1 extends straightforwardly to the algebra of A_{31} . We conclude that the truncation procedure must always result in the same real form of A_{31} , and cannot depend on the choice of A_{10} subalgebra of E_{11} . Hence the groups $SU(32)$, $SL(32, \mathbb{R})$, $SU^*(32)$ and $SU(16, 16)$ cannot be specific to the theories for which we have computed them, but must be the same for all the theories connected by the Weyl group. This assertion can of course be verified by explicit computations.

6 Discussion and conclusions

The E_{11} conjecture asserts that M -theory has a non-linearly realized E_{11} symmetry [10], described by the coset E_{11}/H_{11} . The theory supposedly can be reconstructed from a level expansion, based on a regular A_{10} sub-algebra [15, 16]. The level 0 generators are identified with the elf-bein $GL(11, R)/SO(1, 10)$ of 11 dimensional general relativity. Incorporating space-times with non-Euclidean signatures amounts to introducing signs in H_{11} . There is one more sign that can be adjusted, which is the sign in front of the 4-form term. Starting from the conventional values for M -theory, signature $(1, 10)$ and a conventional sign for the 4-form term, we have argued that the same theory should describe Hull's M^* - and M' -theories [4], essentially by choosing a different A_{10} subalgebra. The sign in front of the 4-form term in the Lagrangian found by Hull exactly agrees with the one we find from E_{11} -algebra. Other signatures than $(1, 10)$, $(2, 9)$, $(5, 6)$, $(6, 5)$, $(9, 2)$ and $(10, 1)$ are not described by this theory: They require a different real form of the denominator sub-algebra H_{11} , and therefore cannot be equivalent. We therefore conclude that the E_{11} -proposal, originally set up to describe M -theory, implicitly includes the M^* - and M' -theories, but no others.

It is satisfying that a computation based on E_{11} reproduces the known theories, and no other ones. Even if not a prediction, the existence of M^* -, and M' -theories can be regarded as a successful, and remarkable postdiction of the conjecture. It therefore seems that a formalism built upon a non-linearly realized E_{11} -symmetry has the potential to provide a framework in which M -, M^* - and M' -theory are treated as a single theory, as already suggested by Hull. It would be interesting to see if similar results could be extracted from other proposals for a more fundamental description of M -theory [30, 31, 32, 33]. A superalgebra perspective on theories with multiple time directions was offered in [34, 35].

We furthermore note that, although it was originally argued that these theories are connected by time-like T-duality, in our arguments closed time-like curves, and the doubts that one may have about them never appear. In the E_{11} -context the duality simply follows from the fact that an arbitrary choice of “gravitational sub-algebra” A_{10} does not guarantee the space-time signature to be $(1, 10)$. If one nevertheless insists on such a space-time signature E_{11} invariance must be broken. It is easily shown that discarding exotic space-time signatures, we must restrict to generators corresponding to roots orthogonal to ω_1 , which breaks the symmetry to E_{10} , and places time back in the special position that it had lost since the advent of Relativity.

As a side-remark, we note that the question of space-time signature can be posed and answered without any resolution on the question of how space-time, and in particular space-time translation symmetries should be represented in the algebra [16, 29, 12, 36]. The possibility to add signs for time-like directions may however be useful as a test for proposals of higher level representations as derivatives of lower level fields [16, 12].

The techniques that were applied here to E_{11} , can also be applied to theories that are dimensionally reduced over a number of directions that includes at least one time-like one. Applied to the coset symmetries of dimensionally reduced supergravity, they elucidate the full duality web, and reveal duality groups that have not appeared in the literature before. A simple link to the theory of real forms of algebra’s makes it an almost trivial exercise to classify them [19]. They can also be applied to other theories based on triple-extended algebra’s [17, 12], where it is easy to see that the duality pattern, and the possible space-time signatures are very dependent on the algebra. We intend to report on these topics in the future.

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